Well-posedness of Backward Stochastic Differential Equations with General Filtration

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April 5, 2011

Abstract

This paper is addressed to the well-posedness of some linear and semilinear backward stochastic differential equations with general filtration, without using the Martingale Representation Theorem. The point of our approach is to introduce a new notion of solution, i.e., the transposition solution, which coincides with the usual strong solution when the filtration is natural but it is more flexible for the general filtration than the existing notion of solutions. A comparison theorem for transposition solutions is also presented.

2010 Mathematics Subject Classification. Primary 60H10; Secondary 34F05, 93E20.

Key Words Backward stochastic differential equations, transposition solution, filtration, comparison theorem.

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1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, on which a 1-dimensional standard Brownian motion $\{w(t)\}_{t\in[0,T]}$ is defined. We denote by $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ ($n \in \mathbb{N}$) the Hilbert space consisting of all $\mathcal{F}_t$-measurable ($\mathbb{R}^n$-valued) random variables $\xi : \Omega \to \mathbb{R}^n$ such that $\mathbb{E}|\xi|^2_{\mathbb{R}^n} < \infty$, with the canonical inner product; by $L^2_{\mathbb{F}}(\Omega; L^r(0,T;\mathbb{R}^n))$ $(1 \le r \le \infty)$ the Banach space consisting of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted stochastic processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^r(0,T;\mathbb{R}^n)}) < \infty$, with the canonical norm; by $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ the Banach space consisting of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n)}) < \infty$, with the canonical norm; by $\mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n)$ the Hilbert space consisting of all $\mathbb{R}^n$-valued square integrable $\{\mathcal{F}_t\}$-martingales, with the canonical inner product; and by $\mathcal{M}^2_{0,\mathbb{F}}([0,T];\mathbb{R}^n)$ the closed subspace $\{X(\cdot) \in \mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n) \mid X(0) = 0 \text{ a.s.}\}$ of $\mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n)$ with the inherited topology. Also, we denote by $D([0,T];\mathbb{R}^n)$ the Banach space of all càdlàg (i.e., right continuous with left limits) functions from $[0,T]$ to $\mathbb{R}^n$, endowed with the inherited topology from $L^\infty(0,T;\mathbb{R}^n)$ rather than the Skorokhod topology; and by $L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n))$ the Banach space consisting of all $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted càdlàg processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n)}) < \infty$, with the canonical norm. For any $t \in [0,T]$, one can define the spaces $L^2_{\mathbb{F}}(\Omega; L^r(t,T;\mathbb{R}^n))$, $L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$, $L^2_{\mathbb{F}}(\Omega; D([t,T];\mathbb{R}^n))$ and so on in a similar way. Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^n$.

This paper is devoted to a study of the well-posedness for the following semilinear backward stochastic differential equation (BSDE for short)

$$\begin{cases} dy(t) = f(t, y(t), Y(t))dt + Y(t)dw(t) & \text{in } [0,T], \\ y(T) = y_T, \end{cases} \tag{1.1}$$

where $y_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, $f(\cdot,\cdot,\cdot)$ satisfies $f(\cdot,0,0) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n))$ and, for some constant $K > 0$,

$$|f(t,p_1,q_1) - f(t,p_2,q_2)| \le K(|p_1 - p_2| + |q_1 - q_2|), \quad t \in [0,T], \text{ a.s.}, \forall\, p_1,p_2,q_1,q_2 \in \mathbb{R}^n. \tag{1.2}$$

(Clearly, one can consider similarly the general case that the term $Y(t)dw(t)$ in (1.1) is replaced by $[g(t,y(t)) + Y(t)]\,dw(t)$ provided that $g(\cdot,0) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ and $g(\cdot,\cdot)$ is globally Lipschitz continuous with respect to its second argument).

The study of BSDEs is stimulated by the classical works [1, 2, 12]. Now, it is well-known that BSDEs and its various variants play important and fundamental roles in Stochastic Control ([13, 18]), Mathematical Finance ([3, 6, 16]), Probability and Stochastic Analysis ([15]), Partial Differential Equations ([11, 14, 16]) and so on.

When $\mathbb{F}$ is equal to the natural filtration $\mathbb{W}$ (generated by the Brownian motion $\{w(\cdot)\}$ and augmented by all the $\mathbb{P}$-null sets), the well-posedness of equation (1.1) is well understood ([12]). In this case, by definition, $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ is said to be a (strong) solution to equation (1.1) if

$$y(t) = y_T - \int_t^T f(s, y(s), Y(s))ds - \int_t^T Y(s)dw(s), \qquad \forall\, t \in [0,T]. \tag{1.3}$$

Clearly, the first step to establish the well-posedness of the semilinear equation (1.1) is to study the same problem but for the following linear BSDE with a non-homonomous term $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n))$:

$$\begin{cases} dy(t) = f(t)dt + Y(t)dw(t), \quad t \in [0,T), \\ y(T) = y_T. \end{cases} \tag{1.4}$$

The main idea in [12] for solving equation (1.1) with $\mathbb{F} = \mathbb{W}$ is as follows: First, for (1.4), noting that the following process

$$M(t) = \mathbb{E}\Big(y_T - \int_0^T f(s)ds \;\Big|\; \mathcal{F}_t\Big) \tag{1.5}$$

is a $\{\mathcal{F}_t\}$-martingale, and using the Martingale Representation Theorem (valid only for the case $\mathbb{F} = \mathbb{W}$), one can find a $Y(\cdot) \in L^2_{\mathbb{W}}(\Omega; L^2(0,T;\mathbb{R}^n))$ such that

$$M(t) = M(0) + \int_0^t Y(s)dw(s). \tag{1.6}$$

Putting

$$y(t) = M(t) + \int_0^t f(s)ds, \tag{1.7}$$

one then finds the unique strong solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{W}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{W}}(\Omega; L^2(0,T;\mathbb{R}^n))$ for the linear BSDE (1.4). Based on this and using the Picard iteration argument, the desired well-posedness for equation (1.1) follows.

It is easy to see that the Martingale Representation Theorem plays a crucial role for the above mentioned well-posedness result for equation (1.1) with natural filtration. In the general case when the filtration $\mathbb{F}$ is not equal to the natural one, $\mathbb{W}$ might be a proper sub-class of $\mathbb{F}$, and therefore, the Martingale Representation Theorem fails. As far as we know, there exists only a very few works addressing the well-posedness of equation (1.1) with the general filtration ([5, 7]).

The main idea to study the well-posedness of BSDEs in [5] is as follows. Consider first equation (1.4). Since the filtration $\mathbb{F}$ is not equal to the natural one, the following

$$\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n) \triangleq \Big\{ \int_0^\cdot g(s)dw(s) \;\Big|\; g(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n)) \Big\} \tag{1.8}$$

is a proper subspace of $\mathcal{M}^2_{0,\mathbb{F}}([0,T];\mathbb{R}^n)$. Then one has the following (unique) orthogonal decomposition:

$$M(\cdot) - M(0) = P(\cdot) + Q(\cdot), \tag{1.9}$$

for some $P(\cdot) \in \mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)$ and $Q(\cdot) \in \Big(\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\Big)^\perp$. By (1.8), there is a $Y(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ such that

$$P(t) = \int_0^t Y(s)dw(s). \tag{1.10}$$

Still, we define $y(\cdot)$ as in (1.7). It is easy to check that $(y(\cdot), Q(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times \Big(\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\Big)^\perp \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ is the unique solution of the following equation

$$y(t) = y_T + Q(t) - Q(T) - \int_t^T f(s)ds - \int_t^T Y(s)dw(s), \qquad \forall\, t \in [0,T]. \tag{1.11}$$

This means that (1.11) is another reasonable "modification" of the linear BSDE (1.4) (by adding another corrected term $Q(\cdot)$). Similar to the above, by utilizing the Picard iteration argument, one can study the well-posedness of equation (1.1) (by adding one more corrected term $dQ(t)$ in the right hand side of the first equation in (1.1)). Note that the appearance of this extra term $Q(\cdot)$ makes the rigorous analysis on the properties of $y(\cdot)$ and $Y(\cdot)$ much more complicated than the case of natural filtration. For example, one needs to use some deep results in martingale theory (e.g., [4, Chapter VIII]) to establish the duality relationship (like (1.15) below) between this sort of modified

BSDEs and the usual (forward) stochastic differential equations although it is not difficult to give the desired relationship formally. Meanwhile, one knows very little about $\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)$ (which is actually introduced to replace the use of Martingale Representation Theorem), and therefore, it seems very difficult to "compute" the above $Y(\cdot)$ in (1.10).

In [7], the authors developed another approach to address the well-posedness of BSDEs. The main idea in [7] for solving equation (1.4) (with general filtration) is as follows. Although formula (1.6) does not make sense any more, $M(\cdot)\in\mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n)$ and $y(\cdot)\in L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))$[1] are still well-defined respectively by (1.5) and (1.7), and verifies $M(0)=y(0)$, a.s. Then, it is easy to check that the above $(y(\cdot),M(\cdot))$ is the unique solution of the following equation

$$y(t)=y_T-\int_t^T f(s)ds+M(t)-M(T),\qquad \forall\, t\in[0,T] \tag{1.12}$$

in the solution space

$$\Upsilon\overset{\triangle}{=}\Big\{(h(\cdot),N(\cdot))\in L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))\times\mathcal{M}^2_{\mathbb{F}}([0,T];\mathbb{R}^n)\;\Big|\;N(0)=h(0)\text{ a.s.}\Big\}. \tag{1.13}$$

This means that (1.12) is a reasonable "modification" of the linear BSDE (1.4). Starting from this and using the Picard iteration argument once more, one can study the well-posedness of equation (1.1) (with a suitable modification) (See [7] for more details). This approach does not need to use the Martingale Representation Theorem, either. However, the adjusting term $Y(\cdot)$ in (1.4) (or more generally, in (1.1)) is then suppressed. Note that this term plays a crucial role in some problems, say the Pontryagin-type maximum principle for general stochastic optimal control problems ([13, 18] and the references therein). On the other hand, it seems to be very difficult to give the duality analysis on solutions of equation (1.12) (or the modified version of (1.1)).

In this paper, we shall present a different approach to treat the well-posedness of BSDEs with general filtration. Our idea is as follows. Fixing $t\in[0,T]$, we consider the following linear (forward) stochastic differential equation

$$\begin{cases} dz(\tau)=u(\tau)d\tau+v(\tau)dw(\tau), & \tau\in(t,T],\\ z(t)=\eta. \end{cases} \tag{1.14}$$

It is clear that, for given $u(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))$, $v(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))$ and $\eta\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, equation (1.14) admits a unique strong solution $z(\cdot)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$. Now, if equation (1.1) admits a strong solution $(y(\cdot),Y(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))$ (say, when $\mathbb{F}=\mathbb{W}$), then, applying Itô's formula to $\langle\, z(t),y(t)\,\rangle$, it is easy to check that

$$\begin{aligned}&\mathbb{E}\,\langle\, z(T),y_T\,\rangle-\mathbb{E}\,\langle\,\eta,y(t)\,\rangle\\ &=\mathbb{E}\int_t^T\langle\, z(\tau),f(\tau,y(\tau),Y(\tau))\,\rangle\,d\tau+\mathbb{E}\int_t^T\langle\, u(\tau),y(\tau)\,\rangle\,d\tau+\mathbb{E}\int_t^T\langle\, v(\tau),Y(\tau)\,\rangle\,d\tau.\end{aligned} \tag{1.15}$$

This inspires us to introduce the following new notion for the solution of equation (1.1).

Definition 1.1 *We call $(y(\cdot),Y(\cdot))\in L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ a transposition solution of equation (1.1) if for any $t\in[0,T]$, $u(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))$, $v(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))$ and $\eta\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, identity (1.15) holds.*

[1] In [7], the authors asserted that $y(\cdot)\in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$ (in terms of our notation). But it seems to us that this should be a misprint.

The main purpose of this paper is to show that equation (1.1) is well-posed in the above transposition sense. Clearly, any transposition solution of equation (1.1) coincides with its strong solution whenever the filtration $\mathbb{F}$ is natural. Note that, in the general case, the space for the first component of the solution is chosen to be $L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n))$ rather than $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$. This is quite natural because the filtration $\mathbb{F}$ is assumed only to be right-continuous.

Our approach is motivated by the classical transposition method in solving the non-homogeneous boundary value problems for partial differential equations ([9]) and especially the boundary controllability problem for hyperbolic equations ([8]). On the other hand, one can find a rudiment of our approach at [18, pp. 353–354] though the space for $y(\cdot)$ was chosen to be $L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ and the filtration was assumed to be natural there. The main advantage of our approach consists in the fact that the duality analysis is contained in the definition of solutions, and therefore, we do not need to utilize the deep result in martingale theory to deduce this sort of duality relationship any more, and one can easily deduce a similar comparison theorem for transposition solutions of (1.1) by using almost the same approach as in the case of natural filtration ([6]). Also, it is even easier (and therefore we omit the details) to establish a Pontryagin-type maximum principle for general stochastic optimal control problems than to solve the same problem with the natural filtration ([13, 18]) because, again, the desired duality analysis is contained in the definition of transposition solution. Moreover, by our method, the adjusting term $Y(\cdot)$ is obtained by the standard Riesz Representation Theorem for Hilbert Space, and therefore, one can utilize the theory from Hilbert Spaces to characterize $Y(\cdot)$, or even give a numerical approach for Y (see Remark 3.2) although the detailed analysis is beyond the scope of this paper.

People may be unsatisfied with our definition on the transposition solution of (1.1) because one does not see what equation this solution satisfies. However, starting from our transposition solution of (1.1), one can obtain a corrected form of this equation, i.e., equation (4.10) in Section 4. Then, by introducing suitably a corrected solution of (1.1) (See Definition 4.1), we obtain also a corresponding well-posedness result (See Corollary 4.1).

The rest of this paper is organized as follows. In Section 2, we show some useful preliminary results. Section 3 is addressed to the well-posedness of the linear BSDE (1.4). Then, we prove the well-posedness of the semilinear BSDE (1.1) in Section 4. Finally, in Section 5, we present a comparison theorem for transposition solutions of (1.1) in one dimension.

2 Preliminaries

In this section, we collect some preliminary results which will be useful in the sequel.

Fix any t_1 and t_2 satisfying $0 \le t_2 \le t_1 \le T$. First of all, we need the following Riesz-type Representation Theorem, which is a special case of the known result in [10, Corollary 2.3 and Remark 2.4].

Lemma 2.1 *For any $r \in [1,\infty)$, it holds that*

$$\big(L^2_{\mathbb{F}}(\Omega; L^r(t_2,t_1;\mathbb{R}^n))\big)^* = L^2_{\mathbb{F}}(\Omega; L^{r'}(t_2,t_1;\mathbb{R}^n)),$$

where $r' = r/(r-1)$ if $r \neq 1$; $r' = \infty$ if $r = 1$.

Next, we need the following simple result (whose proof is direct, and therefore we omit the details).

Lemma 2.2 *There is a constant C, depending only on T, such that for any $\big(u(\cdot),v(\cdot),\eta\big)\in L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the solution $z(\cdot)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ of equation (1.14) satisfies*

$$
\begin{aligned}
&|z(\cdot)|_{L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))}\\
&\leq C\left|\big(u(\cdot),v(\cdot),\eta\big)\right|_{L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)},\qquad \forall\, t\in[0,T].
\end{aligned}
\tag{2.1}
$$

Further, we need the following result, which can be seen as a variant of the classical Lebesgue Theorem (on Lebesgue point).

Lemma 2.3 *Assume that $p\in(1,\infty]$, $q=\begin{cases}\frac{p}{p-1} & \text{if } p\in(1,\infty),\\ 1 & \text{if } p=\infty,\end{cases}$ $f_1\in L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))$ and $f_2\in L^q_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))$. Then*

$$
\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\mathbb{E}\,\langle\, f_1(t),f_2(\tau)\,\rangle\,d\tau=\mathbb{E}\,\langle\, f_1(t),f_2(t)\,\rangle,\qquad t\in[0,T]\ \text{a.e.}\tag{2.2}
$$

Proof. We consider the case that $h\to 0+$ (The case that $h\to 0-$ can be considered similarly). Let

$$
\tilde f_2=\begin{cases} f_2, & t\in[0,T]\\ 0, & t\in(T,2T].\end{cases}
$$

Obviously, $\tilde f_2\in L^q_{\mathbb{F}}(0,2T;L^2(\Omega;\mathbb{R}^n))$ and

$$
|\tilde f_2|_{L^q_{\mathbb{F}}(0,2T;L^2(\Omega;\mathbb{R}^n))}=|\tilde f_2|_{L^q_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}=|f_2|_{L^q_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}.
$$

Since $C([0,2T];L^2(\Omega;\mathbb{R}^n))$ is dense in $L^q_{\mathbb{F}}(0,2T;L^2(\Omega;\mathbb{R}^n))$, for any $\varepsilon>0$, one can find $f_2^0\in C([0,2T];L^2(\Omega;\mathbb{R}^n))$ such that

$$
|\tilde f_2-f_2^0|_{L^q_{\mathbb{F}}(0,2T;L^2(\Omega;\mathbb{R}^n))}\leq\varepsilon.\tag{2.3}
$$

By the uniform continuity of $f_2^0(\cdot)$ in $L^2(\Omega;\mathbb{R}^n)$, one can find a $\delta=\delta(\varepsilon)>0$ such that

$$
|f_2^0(s_1)-f_2^0(s_2)|_{L^2(\Omega;\mathbb{R}^n)}\leq\varepsilon,\quad \forall\, s_1,s_2\in[0,2T]\ \text{satisfying}\ |s_1-s_2|\leq\delta.\tag{2.4}
$$

Thanks to (2.4), we see that, when $h\leq\delta$, it holds that

$$
\begin{aligned}
&\int_0^T\left|\frac{1}{h}\int_t^{t+h}\mathbb{E}\,\langle\, f_1(t),f_2^0(\tau)\,\rangle\,d\tau-\mathbb{E}\,\langle\, f_1(t),f_2^0(t)\,\rangle\right|dt\\
&=\frac{1}{h}\int_0^T\left|\int_t^{t+h}\mathbb{E}\,\langle\, f_1(t),f_2^0(\tau)-f_2^0(t)\,\rangle\,d\tau\right|dt\\
&\leq\frac{1}{h}\int_0^T\int_t^{t+h}|f_1(t)|_{L^2(\Omega;\mathbb{R}^n)}|f_2^0(\tau)-f_2^0(t)|_{L^2(\Omega;\mathbb{R}^n)}d\tau dt\\
&\leq\frac{\varepsilon}{h}\int_0^T\int_t^{t+h}|f_1(t)|_{L^2(\Omega;\mathbb{R}^n)}d\tau dt=\varepsilon\int_0^T|f_1(t)|_{L^2(\Omega;\mathbb{R}^n)}dt\leq C\varepsilon|f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}.
\end{aligned}
\tag{2.5}
$$

Also, by (2.3), we have

$$
\begin{aligned}
&\int_0^T\left|\mathbb{E}\,\langle\, f_1(t),\tilde f_2(t)\,\rangle-\mathbb{E}\,\langle\, f_1(t),f_2^0(t)\,\rangle\right|dt\\
&\leq|f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}|\tilde f_2-f_2^0|_{L^q_{\mathbb{F}}(0,2T;L^2(\Omega;\mathbb{R}^n))}\leq\varepsilon|f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}.
\end{aligned}
\tag{2.6}
$$

Further, using (2.3) again, we find

$$
\begin{aligned}
&\int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle \, d\tau - \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), f_2^0(\tau) \rangle \, d\tau \right| dt \\
&= \frac{1}{h} \int_0^T \left| \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) - f_2^0(\tau) \rangle \, d\tau \right| dt \\
&\le \frac{1}{h} \int_0^T \int_t^{t+h} |f_1(t)|_{L^2(\Omega;\mathbb{R}^n)} |\tilde{f}_2(\tau) - f_2^0(\tau)|_{L^2(\Omega;\mathbb{R}^n)} d\tau dt \\
&\le \frac{1}{h} \left[\int_0^T \int_t^{t+h} |f_1(t)|^p_{L^2(\Omega;\mathbb{R}^n)} d\tau dt \right]^{1/p} \left[\int_0^T \int_t^{t+h} |\tilde{f}_2(\tau) - f_2^0(\tau)|^q_{L^2(\Omega;\mathbb{R}^n)} d\tau dt \right]^{1/q} \qquad (2.7) \\
&= |f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))} \left[\frac{1}{h} \int_0^T \int_0^h |\tilde{f}_2(t+\tau) - f_2^0(t+\tau)|^q_{L^2(\Omega;\mathbb{R}^n)} d\tau dt \right]^{1/q} \\
&= |f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))} \left[\frac{1}{h} \int_0^h \int_\tau^{T+\tau} |\tilde{f}_2(t) - f_2^0(t)|^q_{L^2(\Omega;\mathbb{R}^n)} dt d\tau \right]^{1/q} \\
&\le |f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))} \left[\frac{1}{h} \int_0^h \int_0^T |\tilde{f}_2(t) - f_2^0(t)|^q_{L^2(\Omega;\mathbb{R}^n)} dt d\tau \right]^{1/q} \le \varepsilon |f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}.
\end{aligned}
$$

Combining (2.5), (2.6) and (2.7), we conclude that

$$
\int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle \, d\tau - \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle \right| dt \le C\varepsilon |f_1|_{L^p_{\mathbb{F}}(0,T;L^2(\Omega;\mathbb{R}^n))}.
$$

Therefore,

$$
\lim_{h\to 0} \int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle \, d\tau - \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle \right| dt = 0,
$$

which means that

$$
\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle \, d\tau = \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle, \qquad t \in [0,T] \text{ a.e.}
$$

By this and the definition of $\tilde{f}_2(\cdot)$, we conclude that

$$
\begin{aligned}
\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), f_2(\tau) \rangle \, d\tau &= \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle \, d\tau = \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle \\
&= \mathbb{E} \langle f_1(t), f_2(t) \rangle, \qquad t \in [0,T] \text{ a.e.}
\end{aligned}
$$

This completes the proof of Lemma 2.3. □

3 Well-posedness of linear non-homonomous BSDEs

In this section, as a key step to study the well-posedness of the semilinear BSDE (1.1), we consider first the same problem but for equation (1.4). We have the following result.

Theorem 3.1 *For any $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n))$ and any $y_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, system (1.4) admits a unique transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ (in the sense of Definition 1.1). Furthermore, there is a constant C, depending only on T, such that*

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))}\\
&\leq C\Big[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big], \qquad \forall\, t\in[0,T].
\end{aligned}
\tag{3.1}
$$

Proof. We divide the proof into several steps.

Step 1. We define a linear functional ℓ on $L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ as follows:

$$
\begin{aligned}
&\ell\big(u(\cdot), v(\cdot), \eta\big) = \mathbb{E}\,\langle\, z(T), y_T\,\rangle - \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau)\,\rangle\, dt,\\
&\qquad\qquad \forall\, \big(u(\cdot), v(\cdot), \eta\big) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n),
\end{aligned}
$$

where $z(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$ solves equation (1.14).

Using the Hölder inequality and Lemma 2.2, it is easy to show that

$$
\begin{aligned}
&\big|\ell\big(u(\cdot), v(\cdot), \eta\big)\big|\\
&\leq |z(T)|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)} |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)} + |z(\cdot)|_{L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))} |f|_{L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))}\\
&\leq C\Big[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big]\\
&\qquad \times \big|\big(u(\cdot), v(\cdot), \eta\big)\big|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)}, \quad \forall\, t\in[0,T],
\end{aligned}
\tag{3.2}
$$

where $C = C(T)$ is independent of t. From (3.2), we know ℓ is a bounded linear functional on $L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n)) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. Now, by means of Lemma 2.1, we conclude that there exist $y^t(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^\infty(t,T;\mathbb{R}^n))$, $Y^t(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n))$ and $\varsigma^t \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ such that

$$
\begin{aligned}
&\mathbb{E}\,\langle\, z(T), y_T\,\rangle - \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau)\,\rangle\, d\tau\\
&= \mathbb{E}\int_t^T \langle\, u(\tau), y^t(\tau)\,\rangle\, d\tau + \mathbb{E}\int_t^T \langle\, v(\tau), Y^t(\tau)\,\rangle\, d\tau + \mathbb{E}\,\langle\, \eta, \varsigma^t\,\rangle\,.
\end{aligned}
\tag{3.3}
$$

It is clear that $\varsigma^T = y_T$. Furthermore, there is a positive constant $C = C(T)$, independent of t, such that

$$
\begin{aligned}
&|(y^t(\cdot), Y^t(\cdot), \varsigma^t)|_{L^2_{\mathbb{F}}(\Omega;L^\infty(t,T;\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(t,T;\mathbb{R}^n))\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)}\\
&\leq C\Big[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big], \qquad \forall\, t\in[0,T].
\end{aligned}
\tag{3.4}
$$

Step 2. Note that the "solution" $(y^t(\cdot), Y^t(\cdot))$ (obtained in Step 1) may depend on t. In this step, we shall show the time consistency of $(y^t(\cdot), Y^t(\cdot))$, i.e., for any t_1 and t_2 satisfying $0 \leq t_2 \leq t_1 \leq T$, it holds that

$$
\big(y^{t_2}(\tau,\omega), Y^{t_2}(\tau,\omega)\big) = \big(y^{t_1}(\tau,\omega), Y^{t_1}(\tau,\omega)\big), \qquad (\tau,\omega)\in[t_1,T]\times\Omega \ \text{ a.e.} \tag{3.5}
$$

Note that the solution $z(\cdot)$ of equation (1.14) depends on t, and therefore, we also denote it by $z^t(\cdot)$ (whenever there exists a possible confusion). To show $y^{t_2}(\tau,\omega) = y^{t_1}(\tau,\omega)$ for a.e.

$(\tau,\omega)\in[t_1,T]\times\Omega$, we fix any $\varrho(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(t_1,T;\mathbb{R}^n))$ and choose $t=t_1$, $\eta=0$, $v(\cdot)=0$ and $u(\cdot)=\varrho(\cdot)$ in equation (1.14). From (3.3), we see that

$$\mathbb{E}\,\langle\, z^{t_1}(T),y_T\,\rangle-\mathbb{E}\int_{t_1}^T\langle\, z^{t_1}(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varrho(\tau),y^{t_1}(\tau)\,\rangle\,d\tau. \tag{3.6}$$

On the other hand, choosing $t=t_2$, $\eta=0$, $v(\cdot)=0$ and $u(t,\omega)=\chi_{[t_1,T]}(t)\varrho(t,\omega)$ in equation (1.14). It is clear that

$$z^{t_2}(\cdot)=\begin{cases} z^{t_1}(\cdot), & t\in[t_1,T],\\ 0, & t\in[t_2,t_1).\end{cases}$$

In this case, by (3.3), we have

$$\mathbb{E}\,\langle\, z^{t_1}(T),y_T\,\rangle-\mathbb{E}\int_{t_1}^T\langle\, z^{t_1}(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varrho(\tau),y^{t_2}(\tau)\,\rangle\,d\tau. \tag{3.7}$$

From (3.6) and (3.7), we conclude that

$$\mathbb{E}\int_{t_1}^T\langle\,\varrho(\tau),y^{t_1}(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varrho(\tau),y^{t_2}(\tau)\,\rangle\,d\tau,\qquad\forall\,\varrho(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(t_1,T;\mathbb{R}^n)).$$

From this, we see that $y^{t_2}(\tau,\omega)=y^{t_1}(\tau,\omega)$ for $(\tau,\omega)\in[t_1,T]\times\Omega$ a.e.

To show $Y^{t_2}(\tau,\omega)=Y^{t_1}(\tau,\omega)$ for a.e. $(\tau,\omega)\in[t_1,T]\times\Omega$, we fix any $\varsigma(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(t_1,T;\mathbb{R}^n))$ and choose $t=t_1$, $\eta=0$, $u(\cdot)=0$ and $v(\cdot)=\varsigma(\cdot)$ in equation (1.14) (and denote by $\bar z^{t_1}(\cdot)$ the corresponding solution of (1.14)). From (3.3), we see that

$$\mathbb{E}\,\langle\,\bar z^{t_1}(T),y_T\,\rangle-\mathbb{E}\int_{t_1}^T\langle\,\bar z^{t_1}(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varsigma(\tau),Y^{t_1}(\tau)\,\rangle\,d\tau. \tag{3.8}$$

On the other hand, choosing $t=t_2$, $\eta=0$, $u(\cdot)=0$ and $v(t,\omega)=\chi_{[t_1,T]}(t)\varsigma(t,\omega)$ in equation (1.14) (and denote by $\bar z^{t_2}(\cdot)$ the corresponding solution of (1.14)). It is clear that

$$\bar z^{t_2}(\cdot)=\begin{cases} \bar z^{t_1}(\cdot), & t\in[t_1,T],\\ 0, & t\in[t_2,t_1).\end{cases}$$

In this case, by (3.3), we have

$$\mathbb{E}\,\langle\,\bar z^{t_1}(T),y_T\,\rangle-\mathbb{E}\int_{t_1}^T\langle\,\bar z^{t_1}(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varsigma(\tau),Y^{t_2}(\tau)\,\rangle\,d\tau. \tag{3.9}$$

From (3.8) and (3.9), we conclude that

$$\mathbb{E}\int_{t_1}^T\langle\,\varsigma(\tau),Y^{t_1}(\tau)\,\rangle\,d\tau=\mathbb{E}\int_{t_1}^T\langle\,\varsigma(\tau),Y^{t_2}(\tau)\,\rangle\,d\tau,\qquad\forall\,\varsigma(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(t_1,T;\mathbb{R}^n)).$$

From this, we see that $Y^{t_2}(\tau,\omega)=Y^{t_1}(\tau,\omega)$ for $(\tau,\omega)\in[t_1,T]\times\Omega$ a.e. Hence, (3.5) is verified.

Put

$$y(t,\omega)=y^0(t,\omega),\qquad Y(t,\omega)=Y^0(t,\omega),\qquad\forall\,(t,\omega)\in[0,T]\times\Omega. \tag{3.10}$$

Then, in view of (3.5), it follows that

$$\big(y^t(\tau,\omega),Y^t(\tau,\omega)\big)=\big(y(\tau,\omega),Y(\tau,\omega)\big),\qquad(\tau,\omega)\in[t,T]\times\Omega\ \text{ a.e.} \tag{3.11}$$

Combining (3.3) and (3.11), we find that

$$\begin{aligned}&\mathbb{E}\,\langle\, z(T), y_T\,\rangle - \mathbb{E}\,\langle\, \eta, \varsigma^t\,\rangle\\&=\mathbb{E}\int_t^T \langle\, z(\tau), f(\tau)\,\rangle\, d\tau+\mathbb{E}\int_t^T \langle\, u(\tau), y(\tau)\,\rangle\, d\tau+\mathbb{E}\int_t^T \langle\, v(\tau), Y(\tau)\,\rangle\, d\tau.\end{aligned}\tag{3.12}$$

Step 3. We show in this step that ς^t has a càdlàg modification. For this, clearly, it suffices to show that

$$X(t)\triangleq \varsigma^t-\int_0^t f(s)ds,\qquad t\in[0,T]\tag{3.13}$$

is a $\{\mathcal{F}_t\}$-martingale. The rest of this step is to show that $\{X(t)\}$ is a $\{\mathcal{F}_t\}$-martingale.

First of all, we claim that, for each $t\in[0,T]$,

$$\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)=\varsigma^t,\quad \text{a.s.}\tag{3.14}$$

To show this, choosing $z(t)=\varsigma^t$, $u=0$ and $v=0$ in (1.14), it follows that

$$\mathbb{E}\,\langle\,\varsigma^t, y_T\,\rangle - \mathbb{E}|\varsigma^t|^2=\mathbb{E}\int_t^T \langle\,\varsigma^t, f(s)\,\rangle\, ds.$$

This gives

$$\mathbb{E}\,\langle\,\varsigma^t, \mathbb{E}(y_T\mid\mathcal{F}_t)\,\rangle - \mathbb{E}|\varsigma^t|^2=\mathbb{E}\Big\langle\varsigma^t,\mathbb{E}\Big(\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big\rangle.\tag{3.15}$$

From equality (3.15), we have

$$\mathbb{E}\Big\langle\varsigma^t,\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big\rangle=\mathbb{E}|\varsigma^t|^2.\tag{3.16}$$

On the other hand, choosing $z(t)=\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)$, $u=0$ and $v=0$ in (1.14), we obtain that

$$\begin{aligned}&\mathbb{E}\Big\langle\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big),y_T\Big\rangle-\mathbb{E}\Big\langle\varsigma^t,\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big\rangle\\&=\mathbb{E}\Big\langle\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big),\int_t^T f(s)ds\Big\rangle\\&=\mathbb{E}\Big\langle\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big),\mathbb{E}\Big(\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big\rangle.\end{aligned}\tag{3.17}$$

From equality (3.17), we arrive at

$$\mathbb{E}\Big|\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big|^2-\mathbb{E}\Big\langle\varsigma^t,\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)\Big\rangle=0.\tag{3.18}$$

Combining equality (3.16) and (3.18), we end up with

$$\mathbb{E}\Big|\mathbb{E}\Big(y_T-\int_t^T f(s)ds\;\Big|\;\mathcal{F}_t\Big)-\varsigma^t\Big|^2=0,$$

which gives (3.14).

Next, combining (3.4) and (3.14), it is easy to see that $\varsigma^{\cdot} \in L^2_{\mathbb{F}}(\Omega; L^\infty(0,T;\mathbb{R}^n))$. Hence, $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$.

Now, for any $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 \le \tau_2$, by (3.14), it follows that

$$\begin{aligned}
\mathbb{E}(X(\tau_2) \mid \mathcal{F}_{\tau_1}) &= \mathbb{E}\Big(\varsigma^{\tau_2} - \int_0^{\tau_2} f(s)ds \;\Big|\; \mathcal{F}_{\tau_1}\Big) \\
&= \mathbb{E}\left[\mathbb{E}\Big(y_T - \int_{\tau_2}^T f(s)ds \;\Big|\; \mathcal{F}_{\tau_2}\Big) - \int_0^{\tau_2} f(s)ds \;\Big|\; \mathcal{F}_{\tau_1}\right] \\
&= \mathbb{E}\Big(y_T - \int_0^T f(s)ds \;\Big|\; \mathcal{F}_{\tau_1}\Big) \\
&= \mathbb{E}\Big(y_T - \int_{\tau_1}^T f(s)ds \;\Big|\; \mathcal{F}_{\tau_1}\Big) - \int_0^{\tau_1} f(s)ds \\
&= \varsigma^{\tau_1} - \int_0^{\tau_1} f(s)ds \\
&= X(\tau_1), \quad \text{a.s.}
\end{aligned} \tag{3.19}$$

Therefore, $\{X(t)\}_{0\le t\le T}$ is a $\mathcal{F}_t$-martingale.

Step 4. In this step, we show that, for a.e $t \in [0,T]$,

$$\varsigma^t = y(t) \quad \text{a.s.} \tag{3.20}$$

Fix any $\gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega;\mathbb{R}^n)$. Choosing $t = t_2$, $u(\cdot) = 0$, $v(\cdot) = 0$ and $\eta = (t_1 - t_2)\gamma$ in (1.14), using (3.12), we obtain that

$$\mathbb{E}\,\langle (t_1-t_2)\gamma, y_T\,\rangle - \mathbb{E}\,\langle (t_1-t_2)\gamma, \varsigma^{t_2}\,\rangle = \mathbb{E}\int_{t_2}^T \langle (t_1-t_2)\gamma, f(\tau)\,\rangle\, d\tau. \tag{3.21}$$

Choosing $t = t_2$, $u(\tau,\omega) = \chi_{[t_2,t_1]}(\tau)\gamma(\omega)$, $v(\cdot) = 0$ and $\eta = 0$ in (1.14), using (3.12) once more, we conclude that

$$\begin{aligned}
&\mathbb{E}\,\langle (t_1-t_2)\gamma, y_T\,\rangle \\
&= \mathbb{E}\int_{t_2}^{t_1} \langle (\tau-t_2)\gamma, f(\tau)\,\rangle\, d\tau + \mathbb{E}\int_{t_1}^T \langle (t_1-t_2)\gamma, f(\tau)\,\rangle\, d\tau + \mathbb{E}\int_{t_2}^{t_1} \langle \gamma, y(\tau)\,\rangle\, d\tau.
\end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we end up with

$$\mathbb{E}\,\langle \gamma, \varsigma^{t_2}\,\rangle = \frac{1}{t_1-t_2}\mathbb{E}\int_{t_2}^{t_1} \langle (\tau-t_2)\gamma, f(\tau)\,\rangle\, d\tau - \int_{t_2}^{t_1} \langle \gamma, f(\tau)\,\rangle\, d\tau + \frac{1}{t_1-t_2}\int_{t_2}^{t_1} \mathbb{E}\,\langle \gamma, y(\tau)\,\rangle\, d\tau.$$

It is easy to show that

$$\lim_{t_1\to t_2+0} \frac{1}{t_1-t_2}\mathbb{E}\int_{t_2}^{t_1} \langle (\tau-t_2)\gamma, f(\tau)\,\rangle\, d\tau = \lim_{t_1\to t_2+0} \int_{t_2}^{t_1} \langle \gamma, f(\tau)\,\rangle\, d\tau = 0, \qquad \forall\, \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega;\mathbb{R}^n).$$

Hence,

$$\lim_{t_1\to t_2+0} \frac{1}{t_1-t_2}\int_{t_2}^{t_1} \mathbb{E}\,\langle \gamma, y(\tau)\,\rangle\, d\tau = \mathbb{E}\,\langle \gamma, \varsigma^{t_2}\,\rangle, \qquad \forall\, \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega;\mathbb{R}^n). \tag{3.23}$$

Now, we need to compute the limit $\lim_{t_1 \to t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \gamma, y(\tau) \rangle d\tau$ for some special γ. We consider first the simple case that $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ is a separable Hilbert space. In this case, one can find a sequence $\{\gamma_k\}_{k=1}^\infty$ which is dense in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. For each k, by the classical Lebesgue Theorem (on Lebesgue point), we conclude that there is a Lebesgue null set E_k such that

$$\lim_{t_1 \to t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \gamma_k, y(\tau) \rangle d\tau = \langle \gamma_k, y(t_2) \rangle, \qquad \forall\, t_2 \in [0, T] \setminus E_k. \tag{3.24}$$

Put $E = \bigcup_{k=1}^{\infty} E_k$, whose Lebesque measure is 0. By (3.24) and noting the density of $\{\gamma_k\}_{k=1}^\infty$ in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, it follows that

$$\lim_{t_1 \to t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \gamma, y(\tau) \rangle d\tau = \langle \gamma, y(t_2) \rangle, \qquad \forall\, \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; \mathbb{R}^n), \forall\, t_2 \in [0, T] \setminus E. \tag{3.25}$$

Combining (3.23) and (3.25), we find that $\mathbb{E} \langle \gamma, \varsigma^{t_2} \rangle = \mathbb{E} \langle \gamma, y(t_2) \rangle$ for any $\gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; \mathbb{R}^n)$ and any $t_2 \in [0, T] \setminus E$. Hence, $\varsigma^t = y(t)$ in $[0, T] \times \Omega$, a.e.

Now, we analyze the general case that $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ may not be a separable Hilbert space. In this case, by (3.23), we conclude that

$$\lim_{t_1 \to t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \varsigma^{t_2} - y(t_2), y(\tau) \rangle d\tau = \mathbb{E} \langle \varsigma^{t_2} - y(t_2), \varsigma^{t_2} \rangle. \tag{3.26}$$

Using Lemma 2.3, it follows

$$\lim_{t_1 \to t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \varsigma^{t_2} - y(t_2), y(\tau) \rangle d\tau = \mathbb{E} \langle \varsigma^{t_2} - y(t_2), y(t_2) \rangle, \qquad t_2 \in [0, T] \text{ a.e.} \tag{3.27}$$

By (3.26)–(3.27), we arrive at

$$\mathbb{E} \langle \varsigma^{t_2} - y(t_2), \varsigma^{t_2} \rangle = \mathbb{E} \langle \varsigma^{t_2} - y(t_2), y(t_2) \rangle, \qquad t_2 \in [0, T] \text{ a.e.} \tag{3.28}$$

By (3.28), we find that $\mathbb{E} \left| \varsigma^{t_2} - y(t_2) \right|^2 = 0$ for $t_2 \in [0, T]$ a.e., which implies (3.20) immediately.

Finally, combining (3.20) and the result in Step 3 that ς^t has a càdlàg modification, we see that there is a càdlàg $\mathbb{R}^n$-valued process $\{\tilde{y}(t)\}_{t \in [0,T]}$ such that $y(\cdot) = \tilde{y}$ in $[0, T] \times \Omega$ a.e. It is easy to check that $(\tilde{y}(\cdot), Y(\cdot))$ is a transposition solution to equation (1.4). To simplify the notation, we still use y instead of $\tilde{y}$ to denote the first component of the solution. This means that equation (1.4) admits one and only one transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}^n))$, which completes the proof of Theorem 3.1. $\square$

Remark 3.1 *For the linear BSDE (1.4), we may introduce another notion of solution. We call $y(\cdot) \in L^2_{\mathbb{F}}(\Omega; D([0, T]; \mathbb{R}^n))$ a transposition pseudo-solution of equation (1.4) if for any $t \in [0, T]$, $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(t, T; \mathbb{R}^n))$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$, the following identity holds*

$$\mathbb{E} \langle z(T), y_T \rangle - \mathbb{E} \langle \eta, y(t) \rangle = \mathbb{E} \int_t^T \langle z(\tau), f(\tau) \rangle d\tau + \mathbb{E} \int_t^T \langle u(\tau), y(\tau) \rangle d\tau, \tag{3.29}$$

where $z(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$ solves equation (1.14) with $v(\cdot) = 0$. Using almost the same proof as that of Theorem 3.1, one can show that, for any $f(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}^n))$ and any $y_T \in$

$L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, system (1.4) admits a unique transposition pseudo-solution $y(\cdot)\in L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))$. Furthermore, there is a constant C, depending only on T, such that

$$|y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;D([t,T];\mathbb{R}^n))}\le C\left[|f(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n))}+|y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right],\qquad\forall\, t\in[0,T].$$

It is clear that the transposition pseudo-solution $y(\cdot)$ of equation (1.4) coincides with the first component of the the transposition solution $y(\cdot)$ of equation (1.4). Nevertheless, the transposition pseudo-solution is not a good notion for solution of equation (1.4) because it does not reproduce the strong solution even if the filtration is natural.

Remark 3.2 *At least conceptually, we can give a "numerical" approach for BSDEs with the general filtration in terms of the transposition solution. Indeed, let $\{H_m\}_{m=1}^{+\infty}$ be a sequence of subspaces of $L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ such that for any $g(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$, there exists a sequence $\{g_m(\cdot)\}_{m=1}^{+\infty}$ satisfies that*

$$g_m(\cdot)\in H_m \text{ and } \lim_{m\to+\infty}|g_m-g|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))}=0.$$

Now, for a fixed $m\in\mathbb{N}$, choosing $t=0$, $\eta=0$, $u(\cdot)=0$ and $v(\cdot)=v_m(\cdot)\in H_m$ in equation (1.14) (and denote by $\bar z_m(\cdot)$ the corresponding solution of (1.14)). From (3.3), we see that

$$\mathbb{E}\,\langle\,\bar z_m(T),y_T\,\rangle-\mathbb{E}\int_0^T\langle\,\bar z_m(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_0^T\langle\,v_m(\tau),Y(\tau)\,\rangle\,d\tau. \tag{3.30}$$

On the other hand, using the same argument to obtain $Y(\cdot)$ (by Riesz's Representation Theorem), we can find a $Y_m(\cdot)\in H_m$ such that

$$\mathbb{E}\,\langle\,\bar z_m(T),y_T\,\rangle-\mathbb{E}\int_0^T\langle\,\bar z_m(\tau),f(\tau)\,\rangle\,d\tau=\mathbb{E}\int_0^T\langle\,v_m(\tau),Y_m(\tau)\,\rangle\,d\tau. \tag{3.31}$$

This, together with (3.30), *implies that $Y_m(\cdot)$ is the orthogonal projection of $Y(\cdot)$ to H_m. By the definition of H_m, we know that*

$$\lim_{m\to+\infty}|Y_m(\cdot)-Y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))}=0.$$

Therefore, one can get a "good" approximation of $Y(\cdot)$ if one can choose a suitable sequence $\{H_m\}_{m=1}^{+\infty}$ such that $Y_m(\cdot)$ (say, belongs to a finite dimensional space) can be computed efficiently and that $Y_m(\cdot)$ converges to $Y(\cdot)$ in $L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ in some sense. This will be done in our forthcoming work.

Remark 3.3 *It is clear that one of the key observation in the proof of Theorem 3.1 is that the process $X(t)$ defined by (3.13) is a $\mathcal{F}_t$-martingale. Combining this fact and (3.20), we see that the following process*

$$M(t)\overset{\triangle}{=}y(t)-\int_0^t f(s)ds,\qquad\forall\, t\in[0,T] \tag{3.32}$$

is a $\mathcal{F}_t$-martingale as well. From this, it is easy to check that $(y(\cdot),M(\cdot))$ is the unique solution of equation (1.12) in the solution space Υ (defined by (1.13)). Hence, starting from our transposition solution $(y(\cdot),Y(\cdot))$ for the linear BSDE (1.4), one can re-construct the solution $(y(\cdot),M(\cdot))$ introduced in [7], through the relationship (3.32) between $M(\cdot)$ and $y(\cdot)$. Note however that one cannot do the reverse because for the later one needs to represent $Y(\cdot)$ in terms of $M(\cdot)$, which is exactly the concern of the Martingale Representation Theorem. Nevertheless, since the solution $(y(\cdot),M(\cdot))$ of equation (1.12) is unique in Υ, it is easy to see that the first component of this solution coincides with the first component of the transposition solution $(y(\cdot),Y(\cdot))$ for equation (1.4).

Remark 3.4 *From the proof of Theorem 3.1, it is easy to see why we choose the space for the first component of the transposition solution to be $L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n))$ rather than $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ because a $\mathcal{F}_t$-martingale has only a càdlàg modification. Indeed, as far as we know, there is no general solution on the problem: Under what conditions, a martingale has a continuous modification? We refer to [17, Theorem 2.1.44] for partial solution to this problem.*

Before ending this section, we put

$$L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n) \triangleq \left\{ h \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \;\Big|\; \mathbb{E}h = 0 \right\},$$
$$L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n) \triangleq \left\{ \int_0^T g(s)dw(s) \;\Big|\; g(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n)) \right\}.$$

Clearly, $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ is a closed subspace of $L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. Generally, $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ is a proper subspace of $L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. Hence, the orthogonal complement space $\left(L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)\right)^{\perp}$ is well-defined. We have the following result.

Proposition 3.1 *i) If $y_T - \int_0^T f(s)ds - \mathbb{E}\left(y_T - \int_0^T f(s)ds\right) \in L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, then the transposition solution $(y(\cdot),Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ of equation (1.4) is the unique strong solution of this equation, and $y(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$.*

ii) If $y_T - \int_0^T f(s)ds - \mathbb{E}\left(y_T - \int_0^T f(s)ds\right) \in \left(L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)\right)^{\perp}$, then the transposition solution $(y(\cdot),Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ of equation (1.4) is given by the following

$$\begin{cases} y(t) = \mathbb{E}\left(y_T - \int_t^T f(s)ds \;\Big|\; \mathcal{F}_t\right), \\ Y(\cdot) = 0. \end{cases} \tag{3.33}$$

Proof. i) By definition of $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, one can find a $\overline{Y}(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ such that

$$y_T - \int_0^T f(s)ds - \mathbb{E}\left(y_T - \int_0^T f(s)ds\right) = \int_0^T \overline{Y}(s)dw(s).$$

Then, put

$$\bar{y}(t) = \mathbb{E}\left(y_T - \int_0^T f(s)ds\right) + \int_0^t f(s)ds + \int_0^t \overline{Y}(s)dw(s).$$

It is clear that $(\bar{y}(\cdot),\overline{Y}(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ is a strong solution for the linear BSDE (1.4). Clearly, $(\bar{y}(\cdot),\overline{Y}(\cdot))$ is also a transposition solution of (1.4). Hence, by the uniqueness of the transposition solution for (1.4), it follows that $(\bar{y}(\cdot),\overline{Y}(\cdot)) = (y(\cdot),Y(\cdot))$.

ii) Choosing $t = 0$, $\eta = 0$, $u(\cdot) = 0$ and $v(\cdot) = Y(\cdot)$ in equation (1.14), we get $z(\tau) = \int_0^\tau Y(s)dw(s)$. Hence, by definition, (1.15) is now specialized as

$$\mathbb{E}\Big\langle \int_0^T Y(t)dw(t), y_T \Big\rangle = \mathbb{E}\int_0^T \Big\langle \int_0^t Y(s)dw(s), f(t)\Big\rangle dt + \mathbb{E}\int_0^T |Y(t)|^2 dt. \tag{3.34}$$

Noting that

$$\begin{aligned}
&\mathbb{E}\int_0^T \Big\langle \int_0^t Y(s)dw(s), f(t)\Big\rangle dt\\
&=\mathbb{E}\int_0^T \Big\langle \int_0^T Y(s)dw(s), f(t)\Big\rangle dt-\mathbb{E}\int_0^T \Big\langle \int_t^T Y(s)dw(s), f(t)\Big\rangle\\
&=\mathbb{E}\Big\langle \int_0^T Y(t)dw(t), \int_0^T f(s)ds\Big\rangle
\end{aligned}$$

and that

$$\mathbb{E}\Big\langle \int_0^T Y(t)dw(t), \mathbb{E}\left(y_T-\int_0^T f(s)ds\right)\Big\rangle=0,$$

we conclude from (3.34) that

$$\mathbb{E}\Big\langle \int_0^T Y(t)dw(t),\ y_T-\int_0^T f(s)ds-\mathbb{E}\left(y_T-\int_0^T f(s)ds\right)\Big\rangle=\mathbb{E}\int_0^T |Y(t)|^2dt. \tag{3.35}$$

Now by our assumption that $y_T-\displaystyle\int_0^T f(s)ds-\mathbb{E}\left(y_T-\int_0^T f(s)ds\right)\in\big(L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)\big)^{\perp}$, it follows from (3.35) that $Y(\cdot)=0$.

Next, choosing $t=0$, $\eta=0$ and $v(\cdot)=0$, and $u(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))$ (arbitrarily) in equation (1.14), we get $z(\tau)=\displaystyle\int_0^\tau u(s)ds$. Hence, by definition, (1.15) is now specialized as

$$\mathbb{E}\Big\langle \int_0^T u(t)dt, y_T\Big\rangle=\mathbb{E}\int_0^T \Big\langle \int_0^t u(s)ds, f(t)\Big\rangle dt+\mathbb{E}\int_0^T \langle u(t),y(t)\rangle dt.$$

Hence,

$$\mathbb{E}\int_0^T \Big\langle u(t),\ y(t)-y_T+\int_t^T u(s)ds\Big\rangle dt=0.$$

This gives

$$\mathbb{E}\int_0^T \Big\langle u(t),\ y(t)-\mathbb{E}\left(y_T-\int_t^T f(s)ds\ \Big|\ \mathcal{F}_t\right)\Big\rangle dt=0,\quad \forall\, u(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)). \tag{3.36}$$

Now, the first equality in (3.33) follows from (3.36). This completes the proof of Proposition 3.1.

$\square$

Remark 3.5 *Proposition 3.1 ii) justifies our transposition solution. Indeed, when* $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ *is a proper subspace of* $L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ *and* $y_T-\displaystyle\int_0^T f(s)ds-\mathbb{E}\left(y_T-\int_0^T f(s)ds\right)\in\big(L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)\big)^{\perp}$, *it is easy to show that the transposition solution (3.33) is NOT a strong solution of equation (1.4).*

Remark 3.6 *As far as we know, there exists no any satisfactory characterization on* $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. *Especially, it seems to us that it is not very clear when* $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ *is a proper subspace of* $L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. *Of course, it is easy to see that* $L^2_{\mathbb{W},\mathcal{F}_T}(\Omega;\mathbb{R}^n)=L^2_{0,\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ *implies that the Martingale Representation Theorem holds.*

4 Well-posedness of semilinear BSDEs

The purpose of this section is to establish the following well-posedness result for the semilinear BSDE (1.1).

Theorem 4.1 *For any given $y_T \in L^2_{\mathcal{F}_T}(\Omega)$, equation (1.1) admits a unique transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$. Furthermore, there is a constant $C > 0$, depending only on K and T, such that*

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([0,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))}\\
&\leq C\Big[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big].
\end{aligned}
\tag{4.1}
$$

Proof. Fix any $T_1 \in [0,T]$. For any $(p(\cdot), P(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([T_1,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(T_1,T;\mathbb{R}^n))$, we consider the following equation:

$$
\begin{cases}
dy = f(t,p(t),P(t))dt + Ydw & \text{in } [T_1,T],\\
y(T) = y_T.
\end{cases}
\tag{4.2}
$$

By condition (1.2) and Theorem 3.1, equation (4.2) admits a transposition solution $(y(\cdot), Y(\cdot)) \in L^2_{\mathbb{F}}(\Omega;\ D([T_1,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(T_1,T;\mathbb{R}^n))$. This defines a map F from $L^2_{\mathbb{F}}(\Omega; D([T_1,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega;\ L^2(T_1,T;\mathbb{R}^n))$ into itself by $F(p(\cdot),P(\cdot)) = (y(\cdot),Y(\cdot))$.

We claim that the map F is contractive provided that $T-T_1$ is small enough. Indeed, for another $(\hat{p}(\cdot),\hat{P}(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([T_1,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; L^2(T_1,T;\mathbb{R}^n))$, we define $(\hat{y}(\cdot),\hat{Y}(\cdot)) = F(\hat{p}(\cdot),\hat{P}(\cdot))$. Put

$$
\tilde{y}(\cdot) = y(\cdot) - \hat{y}(\cdot), \quad \widetilde{Y}(\cdot) = Y(\cdot) - \hat{Y}(\cdot), \quad \tilde{f}(\cdot) = f(\cdot,p(\cdot),P(\cdot)) - f(\cdot,\hat{p}(\cdot),\hat{P}(\cdot)).
$$

Clearly, $(\tilde{y}(\cdot),\widetilde{Y}(\cdot))$ solves the following equation

$$
\begin{cases}
d\tilde{y} = \tilde{f}(t)dt + \widetilde{Y}dw & \text{in } [T_1,T],\\
\tilde{y}(T) = 0.
\end{cases}
\tag{4.3}
$$

By condition (1.2), it is easy to see that $\tilde{f}(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(T_1,T;\mathbb{R}^n))$ and

$$
\begin{aligned}
&|\tilde{f}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}\\
&\leq K\Big[|p(\cdot)-\hat{p}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))} + |P(\cdot)-\hat{P}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}\Big]\\
&\leq K\big(T-T_1+\sqrt{T-T_1}\big)\Big[|p(\cdot)-\hat{p}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))} + |P(\cdot)-\hat{P}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\Big].
\end{aligned}
\tag{4.4}
$$

Applying Theorem 3.1 to equation (4.3) and noting (4.4), it follows that there is a constant C, depending only on T, such that

$$
\begin{aligned}
&|(\tilde{y}(\cdot),\tilde{Y}(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))} \leq C|\tilde{f}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\\
&\leq CK\big(T-T_1+\sqrt{T-T_1}\big)\Big[|p(\cdot)-\hat{p}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))} + |P(\cdot)-\hat{P}(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\Big].
\end{aligned}
\tag{4.5}
$$

One may choose T_1 so that $CK\big(T-T_1+\sqrt{T-T_1}\big) < 1$, and hence F is a contractive map.

By the Banach fixed point theorem, F has a fixed point $(y(\cdot),Y(\cdot))\in L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))$. It is clear that $(y(\cdot),Y(\cdot))$ is a transposition solution to the following equation:

$$\begin{cases} dy=f(t,y(t),Y(t))dt+Ydw \quad \text{ in } [T_1,T],\\ y(T)=y_T. \end{cases} \tag{4.6}$$

Using again condition (1.2) and similar to (4.4), we see that $f(\cdot,y(\cdot),Y(\cdot))\in L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))$ and

$$\begin{aligned} &|f(\cdot,y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}\\ &\leq |f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}+K\Big[|y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}+|Y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}\Big]\\ &\leq |f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}+K\big(T-T_1+\sqrt{T-T_1}\big)\Big[|y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))}+|Y(\cdot)|_{L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\Big]. \end{aligned} \tag{4.7}$$

Applying Theorem 3.1 to equation (4.6) and noting (4.7), we find that

$$\begin{aligned} &|(y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\\ &\leq C\Big[|f(\cdot,y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}+|y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big]\\ &\leq C\Big[K\big(T-T_1+\sqrt{T-T_1}\big)|(y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\\ &\qquad +|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(T_1,T;\mathbb{R}^n))}+|y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big]. \end{aligned} \tag{4.8}$$

Noting that $K\big(T-T_1+\sqrt{T-T_1}\big)<1$, by (4.8), we get

$$|(y(\cdot),Y(\cdot))|_{L^2_{\mathbb{F}}(\Omega;D([T_1,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(\Omega;L^2(T_1,T;\mathbb{R}^n))}\leq C\Big[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))}+|y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\Big]. \tag{4.9}$$

Repeating the above argument step by step, we obtain the transposition solution of equation (1.1) on $[0,T]$. The uniqueness of this solution is obvious. The desired estimate (4.1) follows from (4.9). This completes the proof of Theorem 4.1. $\square$

Remark 4.1 *For the transposition solution $(y(\cdot),Y(\cdot))$ to equation (1.1), put*

$$M(t)=y(t)-\int_0^t f(s,y(s),Y(s))ds,\qquad \forall\, t\in[0,T]. \tag{4.10}$$

Thanks to Remark 3.3, it is easy to see that $M(\cdot)$ is a $\mathcal{F}_t$-martingale, and $(y(\cdot),M(\cdot),Y(\cdot))$ satisfies the following equation

$$y(t)=y_T-\int_t^T f(s,y(s),Y(s))ds+M(t)-M(T),\qquad \forall\, t\in[0,T]. \tag{4.11}$$

Equation (4.11) can be regarded as a corrected form of equation (1.1).

Stimulating by Remark 4.1, we introduce the following notion for solution of equation (1.1).

Definition 4.1 *We call $(y(\cdot),M(\cdot),Y(\cdot))\in\Upsilon\times L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))$ to be a corrected solution of equation (1.1) if $(y(\cdot),Y(\cdot))$ is a transposition solution of this equation, and (4.11) holds.*

As a consequence of Theorem 4.1 and Remark 4.1, it is easy to prove the following result.

Corollary 4.1 *For any given $y_T \in L^2_{\mathcal{F}_T}(\Omega)$, equation (1.1) admits a unique corrected solution $(y(\cdot), M(\cdot), Y(\cdot)) \in \Upsilon \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$. Furthermore, there is a constant $C > 0$, depending only on K and T, such that*

$$|(y(\cdot), M(\cdot), Y(\cdot))|_{\Upsilon \times L^2_{\mathbb{F}}(\Omega;L^2(0,T;\mathbb{R}^n))} \leq C\left[|f(\cdot,0,0)|_{L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n))} + |y_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)}\right].$$

Remark 4.2 *Clearly, for the corrected solution $(y(\cdot), M(\cdot), Y(\cdot)) \in \Upsilon \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ of equation (1.1) obtained in Corollary 4.1, the first two components satisfy (4.10). Furthermore, if the filtration $\mathbb{F}$ is the natural one, then the last two components of this solution satisfies (1.6).*

Remark 4.3 *Using the method developed in [5], one can find a unique solution $(\tilde{y}(\cdot), \widetilde{Q}(\cdot), \widetilde{Y}(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0,T];\mathbb{R}^n)) \times \left(\mathcal{M}^2_{0,\mathbb{M},\mathbb{F}}([0,T];\mathbb{R}^n)\right)^{\perp} \times L^2_{\mathbb{F}}(\Omega; L^2(0,T;\mathbb{R}^n))$ satisfying the following equation*

$$\tilde{y}(t) = y_T + \widetilde{Q}(t) - \widetilde{Q}(T) - \int_t^T f(s, \tilde{y}(s), \widetilde{Y}(s))ds - \int_t^T \widetilde{Y}(s)dw(s), \qquad \forall\, t \in [0,T]. \tag{4.12}$$

Equation (4.12) can be regarded as another corrected form of equation (1.1). Using Itô's formula to $\langle\, z(t), \tilde{y}(t)\,\rangle$ (Recall (1.14) for $z(\cdot)$), and noting the strong orthogonality of the martingales $\widetilde{Q}(\cdot)$ and $\int_0^{\cdot} \widetilde{Y}(s)dw(s)$ (which follows from some deep results in martingale theory, e.g., [4, Chapter VIII]), one can show that

$$\begin{aligned}&\mathbb{E}\,\langle\, z(T), y_T\,\rangle - \mathbb{E}\,\langle\, \eta, \tilde{y}(t)\,\rangle \\ &= \mathbb{E}\int_t^T \langle\, z(\tau), f(\tau, \tilde{y}(\tau), \widetilde{Y}(\tau))\,\rangle\, d\tau + \mathbb{E}\int_t^T \langle\, u(\tau), \tilde{y}(\tau)\,\rangle\, d\tau + \mathbb{E}\int_t^T \langle\, v(\tau), \widetilde{Y}(\tau)\,\rangle\, d\tau\end{aligned} \tag{4.13}$$

holds for all $t \in [0,T]$, $u(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(t,T;\mathbb{R}^n))$, $v(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^2(t,T;\mathbb{R}^n))$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. Hence, $(\tilde{y}(\cdot), \widetilde{Y}(\cdot))$ is also a transposition solution of (1.1). By the uniqueness result in Corollary 4.1, we conclude that

$$(y(\cdot), M(\cdot), Y(\cdot)) = (\tilde{y}(\cdot), M(0) + \int_0^{\cdot} \widetilde{Y}(s)dw(s) + \widetilde{Q}(\cdot), \widetilde{Y}(\cdot)).$$

Nevertheless, as we explained before, in some sense, our method seems to be more flexible for the general filtration than the existing ones.

Remark 4.4 *In some sense, the transposition/corrected solution for BSDEs is in spirit close to the distribution solution for partial differential equations. It is then very natural to study the further regularity for the transposition solution $(y(\cdot), Y(\cdot))$ (or corrected solution $(y(\cdot), M(\cdot), Y(\cdot))$) of equation (1.1).*

5 Comparison theorem for transposition solutions

In this section, we show a comparison theorem for transposition solutions of the semilinear BSDE (1.1) in one dimension, i.e., $n = 1$.

We will go a little further. Besides equation (1.1) (with $n = 1$), we consider also the following BSDE:

$$\begin{cases} d\bar{y}(t) = \bar{f}(t, \bar{y}(t), \overline{Y}(t))dt + \overline{Y}(t)dw(t) \quad \text{in } [0, T], \\ \bar{y}(T) = \bar{y}_T. \end{cases} \tag{5.1}$$

Here $\bar{y}_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, $\bar{f}(\cdot, \cdot, \cdot)$ is supposed to satisfy $\bar{f}(\cdot, 0, 0) \in L^2_{\mathbb{F}}(\Omega; L^1(0, T; \mathbb{R}))$ and,

$$|\bar{f}(t, p_1, q_1) - \bar{f}(t, p_2, q_2)| \le K(|p_1 - p_2| + |q_1 - q_2|), \quad t \in [0, T] \text{ a.s.}, \forall\, p_1, p_2, q_1, q_2 \in \mathbb{R}. \tag{5.2}$$

By Theorem 4.1, equation (5.1) admits a unique transposition solution $(\bar{y}(\cdot), \overline{Y}(\cdot)) \in L^2_{\mathbb{F}}(\Omega; D([0, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2(0, T; \mathbb{R}))$.

We have the following result.

Theorem 5.1 *If $y_T \ge \bar{y}_T$ a.s., and for any $a, b \in \mathbb{R}$, and a.e. $t \in [0, T]$,*

$$f(t, a, b) \le \bar{f}(t, a, b) \quad a.s., \tag{5.3}$$

then, for any $t \in [0, T]$,

$$y(t) \ge \bar{y}(t) \quad a.s. \tag{5.4}$$

Moreover, $y(t) = \bar{y}(t)$ a.s., for some $t \in [0, T]$ if and only if $y_T = \bar{y}_T$ a.s., and that $f(s, \bar{y}(s), \overline{Y}(s)) = \bar{f}(s, \bar{y}(s), \overline{Y}(s))$ a.s. for a.e. $s \in [t, T]$.

Proof. The idea of our proof is very close to that of [6, Theorem 2.2]. Put $\hat{y} = y - \bar{y}$ and $\hat{Y} = Y - \overline{Y}$. It is clear that $(\hat{y}, \hat{Y})$ is a transposition solution of the following equation

$$\begin{cases} d\hat{y}(t) = (a(t)\hat{y}(t) + b(t)\hat{Y}(t) + h(t))dt + \hat{Y}(t)dw(t) \quad \text{in } [0, T], \\ \hat{y}(T) = y_T - \bar{y}_T, \end{cases} \tag{5.5}$$

where

$$a(t) = \begin{cases} \dfrac{f(t, y(t), Y(t)) - f(t, \bar{y}(t), Y(t))}{y(t) - \bar{y}(t)}, & y(t) \ne \bar{y}(t), \\ 0, & y(t) = \bar{y}(t), \end{cases}$$

$$b(t) = \begin{cases} \dfrac{f(t, \bar{y}(t), Y(t)) - f(t, \bar{y}(t), \overline{Y}(t))}{Y(t) - \overline{Y}(t)}, & Y(t) \ne \overline{Y}(t), \\ 0, & Y(t) = \overline{Y}(t), \end{cases}$$

and

$$h(t) = f(t, \bar{y}(t), \overline{Y}(t)) - \bar{f}(t, \bar{y}(t), \overline{Y}(t)) \le 0.$$

From (1.2) and (5.2), we see that $|a(t)| \le K$ and $|b(t)| \le K$ a.s., for a.e. $t \in [0, T]$.

Now, for any $t \in [0, T]$, we consider the following (forward) stochastic differential equation

$$\begin{cases} dq(s) = -a(s)q(s)dt - b(s)q(s)dw(s) \quad \text{in } [t, T], \\ q(t) = \varsigma, \end{cases} \tag{5.6}$$

where $\varsigma \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$ satisfying $\varsigma \geq 0$ a.s. It is easy to see that

$$q(s) = \varsigma \exp\Big\{ -\int_t^s a(\tau)d\tau - \frac{1}{2}\int_t^s b(\tau)d\tau - \int_t^s b(\tau)dw(\tau)\Big\} \geq 0. \tag{5.7}$$

Since $(\hat{y},\hat{Y})$ is the transposition solution of equation (5.5), by Definition 1.1, it follows that

$$\begin{aligned}\mathbb{E}\big(\hat{y}(T)q(T)\big) - \mathbb{E}\big(\hat{y}(t)\varsigma\big) &= \mathbb{E}\int_t^T [a(s)\hat{y}(s) + b(s)\hat{Y}(s) + h(s)]q(s)ds \\ &\quad -\mathbb{E}\int_t^T \hat{y}(s)a(s)q(s)ds - \mathbb{E}\int_t^T \hat{Y}(s)b(s)q(s)ds,\end{aligned}$$

from which we clonclude that

$$\mathbb{E}\big(\hat{y}(t)\varsigma\big) = \mathbb{E}\big(\hat{y}(T)q(T)\big) - \mathbb{E}\int_t^T h(s)q(s)ds \geq 0, \tag{5.8}$$

for any $\varsigma \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$ such that $\varsigma \geq 0$ a.s. Therefore, we see that $\hat{y}(t) \geq 0$ a.s., which means that $y(t) \geq \bar{y}(t)$ a.s.

Choosing $\varsigma = 1$ in (5.6), from (5.7), it is easy to see that $q(s) > 0$ for any $s \in [t,T]$. By (5.8), we obtain that

$$\mathbb{E}\hat{y}(t) = \mathbb{E}\big(\hat{y}(T)q(T)\big) - \mathbb{E}\int_t^T h(s)q(s)ds \geq 0.$$

If $\hat{y}(t) = 0$ a.s., it follows that $\mathbb{E}\big(\hat{y}(T)q(T)\big) = 0$ and $\mathbb{E}\int_t^T h(s)q(s)ds = 0$. Since $q(s) > 0$ for any $s \in [t,T]$, we have that $\hat{y}(T) = 0$ a.s. and $h(s) = 0$ a.s. for a.e. $s \in [t,T]$, which leads to $y_T = \bar{y}_T$ a.s., and that $f(s,\bar{y}(s),\overline{Y}(s)) = \bar{f}(s,\bar{y}(s),\overline{Y}(s))$ a.s., for a.e. $s \in [t,T]$. $\square$

Acknowledgement

This work is supported by the NSFC under grants 10831007 and 60974035, and the project MTM2008-03541 of the Spanish Ministry of Science and Innovation. The second author acknowledges gratefully Professors Zhenqing Chen, Shige Peng, Jia-An Yan, Jiongmin Yong and Xunyu Zhou for stimulating discussions, and Dr. Mingyu Xu for pointing out reference [7].

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